On the existence of cosmological event horizons

Sourav Bhattacharya and Amitabha Lahiri

S. N. Bose National Centre for Basic Sciences,
Block JD, Sector III, Salt Lake, Kolkata -700098, India.

E-mail: sbhatt@bose.res.in, amitabha@bose.res.in

Abstract. We show that, for general static or axisymmetric stationary spacetimes, a cosmological Killing horizon exists only if $R_{ab}n^a n^b < 0$ for a hypersurface orthogonal timelike n^a, at least over some portion of the region of interest of the manifold. This implies violation of the strong energy condition by the matter fields, the simplest example of which is a positive cosmological constant.

PACS numbers: 04.20.Ha, 04.20.Gz, 95.36.+x

Keywords: Cosmological horizon, strong energy condition, positive cosmological constant

It is generally accepted that a positive cosmological constant implies the existence of a cosmological horizon, i.e. an outer event horizon. If a positive cosmological constant Λ is added into the Einstein equation, we find de Sitter space in the absence of matter for a spatially homogeneous and isotropic universe. This solution exhibits an outer Killing horizon [1]. If the spacetime is assumed to be spherically symmetric and static, or axisymmetric and stationary, the solution to the vacuum Einstein equations is Schwarzschild-de Sitter or Kerr-de Sitter [2]. When the positive Λ and the other parameters of these solutions obey certain conditions between them, one gets stationary black hole space-times embedded within a cosmological event horizon. What happens if there is matter? A sufficiently low matter density should produce a perturbation on the de Sitter black hole background. How does this perturbation affect the global properties of the spacetime? In particular, is there still an outer (cosmological) event horizon? More generally, what is the criterion for the existence of a cosmological event horizon? We were unable to find in the literature anything resembling an existence proof, so we decided to construct one. The motivation to look for horizons in spacetimes with a positive cosmological constant comes from recent observations that our universe is very likely endowed with one [3, 4].

The goal of this paper is to find the general conditions for which a stationary spacetime has an outer cosmological horizon. We consider two types of spacetimes, one static, and the other stationary and axisymmetric. An inner (black hole) event horizon is not assumed, although one may be present. We assume that there is no naked curvature singularity anywhere in our region of interest. This implies that the invariants of the stress-energy tensor are bounded everywhere in our region of interest, and that in the absence of an inner horizon any closed surface can be continuously shrunk to nothing. We also assume that the weak energy condition (WEC) is satisfied by the stress-energy tensor. We assume the existence of a null outer horizon and find

the condition that the stress-energy tensor has to fulfill for the Einstein equations to hold. We find that the strong energy condition must be violated by the stress-energy tensor, at least over some part of the spacelike region inside the outer horizon. While a positive cosmological constant does this, we also find conditions on the stress-energy tensor due to ordinary matter so that $\Lambda > 0$ implies an outer horizon.

Let us then start with a spacetime which is static in some region. In this region the spacetime is endowed with a timelike Killing vector field ξ^a,

$$\nabla_a \xi_b + \nabla_b \xi_a = 0, \tag{1}$$

with norm $\xi_a \xi^a = -\lambda^2$. Since the spacetime is static, ξ^a is orthogonal to a family of spacelike hypersurfaces Σ, and the Frobenius condition is satisfied,

$$\xi_{[a} \nabla_b \xi_{c]} = 0. \tag{2}$$

A horizon of this spacetime is defined as the null hypersurface on which $\lambda^2 = 0$ [5].

Starting from the Killing identity

$$\nabla_a \nabla^a \xi_b = -R_{ab} \xi^a \,, \tag{3}$$

and contracting both sides of Eq. (3) by ξ^b , we obtain

$$\nabla_a \nabla^a \lambda^2 = 2 R_{ab} \xi^a \xi^b - 2 \left(\nabla_a \xi_b\right) \left(\nabla^a \xi^b\right) . \tag{4}$$

On the other hand, we can use Killing's equation (1) and the Frobenius condition (2) to get

$$\nabla_a \xi_b = \frac{1}{\lambda} \left(\xi_b \nabla_a \lambda - \xi_a \nabla_b \lambda\right) , \tag{5}$$

which we substitute into Eq. (4) to obtain

$$\nabla_a \nabla^a \lambda^2 = 2 R_{ab} \xi^a \xi^b + 4 \left(\nabla_a \lambda\right) \left(\nabla^a \lambda\right) . \tag{6}$$

In order to project Eq. (6) onto Σ, we consider the usual projector or the induced metric on Σ

$$h_a{}^b = \delta_a{}^b + \lambda^{-2} \xi_a \xi^b. \tag{7}$$

Let us also write D_a for the induced connection on Σ. Then for any p-form Ω whose projection on Σ is ω, and which satisfies $\pounds_\xi \Omega = 0$ [6],

$$\lambda \nabla_a \Omega^{a\cdots} = D_a (\lambda \omega^{a\cdots}) \,. \tag{8}$$

Choosing the 1-form $\nabla_a \lambda^2$ for Ω_a in this equation, and using Eq. (6), we find

$$D_a \left(\lambda D^a \lambda^2\right) = 2\lambda \left[R_{ab} \xi^a \xi^b + 2 \left(D_a \lambda\right) \left(D^a \lambda\right)\right] \,. \tag{9}$$

We will integrate this equation over the space-like hypersuface with the horizon as boundary. For all known solutions with a horizon defined by $\lambda^2 = 0$, each term on both sides of this equation is finite everywhere on Σ, including on the horizon(s). However, since we have unspecified energy-momentum on Σ, we will not presume that these terms remain finite on the horizons. We will assume instead that the left hand side, for example, does not diverge at the horizon faster than some inverse power of λ. Then we multiply both sides of Eq. (9) by λ^n, for some appropriate $n > 0$, and integrate over the spacelike hypersurface Σ to find

$$\oint_{\partial\Sigma} \lambda^{n+1} D_a \lambda^2 d\gamma^{(2)a} = 2 \int_\Sigma \left[\lambda^{n+1} R_{ab} \xi^a \xi^b + (n + 2\lambda)\, \lambda^n \left(D_a \lambda\right) \left(D^a \lambda\right)\right] . \tag{10}$$

The surface integral is over the boundary of Σ, i.e. the outer horizon whose existence we have assumed, and also over an inner horizon if it exists. On the horizons, $\lambda = 0$.

So if we choose n to be sufficiently large, each of the invariant terms appearing in the right hand side of Eq. (10) remains bounded as $\lambda \to 0$.

Then the surface integral over the horizons vanishes, and we get

$$\int_\Sigma \left[\lambda^{n+1} R_{ab}\xi^a\xi^b + (n+2\lambda)\lambda^n \left(D_a\lambda\right)\left(D^a\lambda\right)\right] = 0. \tag{11}$$

Furthermore, since we have assumed that there is no naked curvature singularity anywhere on Σ, in the absence of an inner black hole horizon we may freely shrink the inner boundary to a non-singular point, so that the corresponding integral vanishes again. Thus Eq. (11) also holds for non-singular spacetimes without a black hole.

The second term in Eq. (11) with a positive n is a spacelike inner product and hence positive definite over Σ, so we must have a negative contribution from the first term $R_{ab}\xi^a\xi^b$. In other words, the outer horizon or the cosmological horizon will exist only if

$$R_{ab}\xi^a\xi^b < 0, \tag{12}$$

at least over some portion of Σ, so that the total integral in Eq. (11) vanishes. Using the Einstein equations

$$R_{ab} - \frac{1}{2}Rg_{ab} = T_{ab}, \tag{13}$$

we see that the condition (12) implies that the strong energy condition (SEC) is violated by the energy-momentum tensor

$$\left(T_{ab} - \frac{1}{2}Tg_{ab}\right)\xi^a\xi^b < 0, \tag{14}$$

at least over some portion of Σ. We know that a positive cosmological constant Λ, appearing on the right hand side of the Einstein equations as $-\Lambda g_{ab}$, violates the SEC. We now split the total stress-energy tensor T_{ab} as

$$T_{ab} = -\Lambda g_{ab} + T^{\mathrm{N}}_{ab}, \tag{15}$$

where the superscript 'N' denotes 'normal' matter fields satisfying the SEC. Then Eq. (11) becomes

$$\int_\Sigma \lambda^n \left[\lambda X^{\mathrm{N}} + (n+2\lambda)(D_a\lambda)(D^a\lambda) - \Lambda\lambda^3\right] = 0. \tag{16}$$

X^{N} is a positive definite contribution from the normal matter satisfying SEC. So for the cosmological horizon to exist, we must have

$$\int_\Sigma \lambda^{n+1}\left[X^{\mathrm{N}} - \Lambda\lambda^2\right] < 0. \tag{17}$$

In other words, the cosmological constant term (with $\Lambda > 0$) has to dominate the integral if there is to be an outer horizon. It is interesting to note that the observed values of Λ and matter densities in the universe satisfy this requirement. So would a universe with $\Lambda > 0$ in which all normal matter is restricted to a finite region in space. This has relevance in discussions of the late time behavior of black holes formed by collapse.

This result can be generalized to stationary axisymmetric spacetimes, in general rotating, which satisfy some additional constraints. The basic scheme will be the same as before. For the spacetime we assume two commuting Killing fields $(\xi^a,\ \phi^a)$,

$$\nabla_{(a}\xi_{b)} = 0 = \nabla_{(a}\phi_{b)}\,, \tag{18}$$

$$[\xi, \phi]^a\ = 0\,. \tag{19}$$

ξ^a is locally timelike with norm $-\lambda^2$, whereas ϕ^a is a locally spacelike Killing field with closed orbits and norm f^2. We also assume that the vectors orthogonal to ξ^a and ϕ^a span an integral submanifold. In other words, local coordinates orthogonal to ξ^a and ϕ^a can be specified everywhere on the spacetime. This, and the last condition above, are the additional constraints mentioned earlier. We note that known stationary axisymmetric spacetimes obey these restrictions.

For a rotating spacetime, ξ^a is not orthogonal to ϕ^a, so in particular there is no spacelike hypersurface tangent to ϕ^a and orthogonal to ξ^a. Let us first construct a family of spacelike hypersurfaces. If we define χ_a as

$$\chi_a = \xi_a - \frac{1}{f^2}\left(\xi_b \phi^b\right)\phi_a \equiv \xi_a + \alpha\phi_a, \tag{20}$$

we will have $\chi_a \phi^a = 0$ everywhere. An orthogonal basis for the spacetime can be written as $\{\chi^a,\ \phi^a,\ \mu^a,\ \nu^a\}$. We note that

$$\chi_a \chi^a = -\beta^2 = -\left(\lambda^2 + \alpha^2 f^2\right), \tag{21}$$

i.e., χ_a is timelike when $\beta^2 > 0$. We can also calculate that

$$\nabla_{(a}\chi_{b)} = \phi_a \nabla_b \alpha + \phi_b \nabla_a \alpha. \tag{22}$$

Our assumption that $\{\mu^a,\ \nu^a\}$ span an integral 2-manifold implies that

$$\chi_{[a}\phi_b \nabla_c \phi_{d]} = 0\,, \tag{23}$$

$$\phi_{[a}\chi_b \nabla_c \chi_{d]} = 0\,. \tag{24}$$

where we have also used Eq. (20). A straightforward calculation from here shows that

$$\nabla_a \chi_b - \nabla_b \chi_a = 2\beta^{-1}\left(\chi_b \nabla_a \beta - \chi_a \nabla_b \beta\right). \tag{25}$$

It follows that χ^a satisfies the Frobenius condition,

$$\chi_{[a}\nabla_b \chi_{c]} = 0, \tag{26}$$

so there is a family of spacelike hypersurfaces Σ orthogonal to χ^a, although we should note that χ^a is not a Killing vector field. In a rotating black hole spacetime, ξ^a becomes spacelike within the ergosphere [7], so for such spacetimes $\lambda^2 = 0$ does not in general define a horizon. The horizons are now located at $\beta^2 = 0$, as we will justify below.

Since χ^a is not a Killing vector — α in Eq. (20) is not a constant — we need to ask if $\beta^2 = 0$ is a Killing horizon, i.e. if there is a Killing vector which becomes null on the surface $\beta^2 = 0$. In order to understand the nature of the surface $\beta^2 = 0$, let us consider the congruence of null geodesics on this surface. We start by constructing a null geodesic on the surface $\beta^2 = 0$.

The normal to a hypersurface defined by $u = 0$ is proportional to $\nabla_a u$. Since the vector field χ^a is hypersurface orthogonal as we have seen in Eq. (26), we can write [8]

$$\chi_a = e^{\rho}\nabla_a u, \tag{27}$$

for some ρ and u.

A straightforward computation using Eq. (22) then shows that, when $\beta^2 = 0$,

$$\chi^a \nabla_a \chi_b = \frac{1}{2}\nabla_b \beta^2 = \kappa \chi_b, \tag{28}$$

where $\kappa := \pounds_\chi \rho$ is a function over that surface. Eq. (28) shows that the 1-form $\nabla_a \beta^2$, which is normal to the $\beta^2 = 0$ surface, is null on that surface. Eq. (28) also yields $\pounds_\chi \kappa = 0$.

Now we can define a null geodesic k^a $\left(\text{i.e.}, k_a k^a = 0;\ k^a\nabla_a k^b = 0\right)$, tangent (or normal) to the surface $\beta^2 = 0$, by $k^a = e^{-\kappa\tau}\chi^a$ [7], where τ is the parameter along χ^a satisfying $\chi^a\nabla_a\tau = 1$. Then using Eq.s (22) and (26) we find that k^a satisfies

$$k_{[a}\nabla_{b]}k_c = e^{-2\kappa\tau}\left[\frac{1}{2}\chi_{(a}\phi_b\nabla_{c)}\alpha - \chi_c\nabla_a\chi_b - \chi_b\phi_a\nabla_c\alpha - \chi_b\phi_c\nabla_a\alpha - \chi_c\chi_{[a}\nabla_{b]}\left(\kappa\tau\right)\right]. \tag{29}$$

We next consider the Raychaudhuri equation for the null geodesic congruence $\{k^a\}$,

$$\frac{d\theta}{d\Theta} = -\frac{1}{2}\theta^2 - \hat{\sigma}_{ab}\hat{\sigma}^{ab} + \hat{\omega}_{ab}\hat{\omega}^{ab} - R_{ab}k^a k^b, \tag{30}$$

where Θ is the parameter along the geodesic k^a; θ, $\hat{\sigma}_{ab}$ and $\hat{\omega}_{ab}$ are respectively the expansion, shear and rotation of the congruence defined by

$$\theta = \hat{h}^{ab}\widehat{\nabla_a k_b}; \qquad \hat{\sigma}_{ab} = \widehat{\nabla_{(a}k_{b)}} - \frac{1}{2}\theta\hat{h}_{ab}; \qquad \hat{\omega}_{ab} = \widehat{\nabla_{[a}k_{b]}}. \tag{31}$$

The 'hat' over the tensors denotes that they are evaluated on a spacelike 2-plane orthogonal to k^a (or $\nabla_a\beta^2$), and $\hat{h}_{ab}$ is the metric on this plane (see e.g. [7, 9] for details on null congruence). Tangent to the $\beta^2 = 0$ surface (i.e., orthogonal to $\nabla_a\beta^2$), we can choose ϕ^a as a basis vector on the spacelike 2-plane. Let the other basis vector be some X^a, with $\phi_a X^a = 0$, and of course $\phi_a k^a = 0 = X_a k^a$. Also, since ϕ^a is a Killing field and commutes with ξ^a, we have $\pounds_\phi\alpha = 0$.

We now contract Eq. (29) by $(\phi^b\phi^c + X^b X^c)$ to find that

$$k_a(\phi^c\phi^b + X^c X^b)\nabla_b k_c = 0\,, \tag{32}$$

which implies the expansion $\theta = 0$ on the surface $\beta^2 = 0$, and thus the left hand side of Eq. (30) is also zero. Similarly by contracting Eq. (29) by $\phi^{[b}X^{c]}$, we see that the rotation $\hat{\omega}_{ab}$ also vanishes. However if we contract Eq. (29) by $\phi^{(b}X^{c)}$, we see that the components of the shear $\hat{\sigma}_{ab}$ do not vanish,

$$k_a\phi^{(b}X^{c)}\hat{\sigma}_{bc} = \frac{1}{2}e^{-\kappa\tau}\phi^{(b}X^{c)}\phi_c\left(\nabla_b\alpha\right)k_a, \tag{33}$$

where we have used the fact that $\theta = 0$. Since the Ricci scalar R is finite at $\beta^2 = 0$ by assumption, Eq. (30) becomes upon using the Einstein equations

$$T_{ab}k^a k^b = -\frac{1}{4}e^{-2\kappa\tau}f^2\left(\widehat{\nabla}_a\alpha\right)\left(\widehat{\nabla}^a\alpha\right). \tag{34}$$

The inner product on the right hand side of Eq. (34) is spacelike. So we see that the null energy condition could be violated on $\beta^2 = 0$,

$$T_{ab}\chi^a\chi^b \leq 0. \tag{35}$$

This also implies by continuity the violation of the WEC close to the hypersurface $\beta^2 = 0$. Since by our assumption we are not violating WEC, we must have $\left(\widehat{\nabla}_a\alpha\right)\left(\widehat{\nabla}^a\alpha\right) = 0$ on the spacelike section of the $\beta^2 = 0$ surface. On the other hand, using Eq. (19) we see that $\pounds_\chi\alpha = 0$ everywhere. Thus α is a constant on the $\beta^2 = 0$ surface. Therefore, on this surface, χ^a coincides with a Killing vector field, and hence the horizons we have defined are Killing horizons. This is actually an old result [10], which we have rederived using an alternative method.

After this necessary digression, we return to our main proof of existence of horizons. Using the Killing identities $\nabla_a\nabla^a\xi_b = -R_b{}^a\xi_a$, and $\nabla_a\nabla^a\phi_b = -R_b{}^a\phi_a$, and also the orthogonality $\chi_a\phi^a = 0$, we obtain on Σ

$$\chi^b\nabla_a\nabla^a\chi_b = -R_{ab}\chi^a\chi^b + 2\chi^a\nabla_c\phi_a\nabla^c\alpha\,, \tag{36}$$

which is equivalent to

$$\nabla_a \nabla^a \beta^2 = 2R_{ab}\chi^a\chi^b - 2\nabla^c\chi^a\nabla_c\chi_a - 4\chi^a\nabla_c\phi_a\nabla^c\alpha. \tag{37}$$

Note that if we set $\alpha = 0$ in Eq. (37), we recover the static case of Eq. (4).

Next we note that the subspace spanned by $\{\chi^a,\ \mu^a,\ \nu^a\}$ do not form a hypersurface. This is because the necessary and sufficient condition that an arbitrary subspace of a manifold forms an integral submanifold or a hypersurface is the existence of a Lie algebra of the basis vectors of that subspace (see e.g. [7] and references therein). The condition Eq. (26) follows from this. On the other hand, Lie brackets among $\{\chi^a,\ \mu^a,\ \nu^a\}$ do not close. For example,

$$[\chi,\ \mu]^a = [\xi,\ \mu]^a + \alpha[\phi,\ \mu]^a + \phi^a\mu^b\nabla_b\alpha. \tag{38}$$

Since μ^a is not a Killing field, the last term on the right hand side of Eq. (38) is not zero. A similar argument holds for ν^a . Therefore the vectors spanned by $\{\chi^a,\ \mu^a,\ \nu^a\}$ do not form a Lie algebra. This implies that we cannot write a condition like $\phi_{[a}\nabla_b\phi_{c]} = 0$.

However, according to our assumptions, there are integral spacelike 2-manifolds orthogonal to both χ^a and ϕ^a. These are spanned by $\{\mu^a,\ \nu^a\}$. Then we must have

$$\phi_{[a}D_b\phi_{c]} = 0, \tag{39}$$

where D_b is the connection induced on Σ defined via the projector $h_a{}^b = \delta_a{}^b + \beta^{-2}\chi_a\chi^b$, exactly in the same manner as in the static case. Then we can write

$$D_a\phi_b = h_a{}^c h_b{}^d \nabla_c\phi_d = \nabla_a\phi_b + \beta^{-2}\left(\chi_a\phi^c\nabla_b\chi_c - \chi_b\phi^c\nabla_a\chi_c\right). \tag{40}$$

Using the expression of $\nabla_a\chi_b$ from Eq. (22) and Eq. (25), we can rewrite this as

$$D_a\phi_b = \nabla_a\phi_b + \frac{f^2}{2\beta^2}\left[\chi_a\nabla_b\alpha - \chi_b\nabla_a\alpha\right]. \tag{41}$$

It follows from this equation that we can write the Killing equation for ϕ_a on Σ as

$$D_a\phi_b + D_b\phi_a = 0. \tag{42}$$

Using this equation and the Frobenius condition of Eq. (39), we derive the expression

$$\nabla_a\phi_b = \frac{1}{f}\left[\phi_b D_a f - \phi_a D_b f\right] + \frac{f^2}{2\beta^2}\left[\chi_b\nabla_a\alpha - \chi_a\nabla_b\alpha\right]. \tag{43}$$

These are all that is needed to simplify Eq. (37). Substituting the expressions for $\nabla_a\chi_b$ and $\nabla_a\phi_b$ into Eq. (37) we get

$$\nabla_a\nabla^a\beta^2 = 2R_{ab}\chi^a\chi^b + 4\left(\nabla_a\beta\right)\left(\nabla^a\beta\right) + f^2\left(\nabla_a\alpha\right)\left(\nabla^a\alpha\right). \tag{44}$$

With this, using the same line of argument as for Eq. (16), we get

$$\int_\Sigma \beta^n \left[\beta X^{\mathrm{N}} + (n+2\beta)\left(D_a\beta\right)\left(D^a\beta\right) + \frac{f^2\beta}{2}\left(D_a\alpha\right)\left(D^a\alpha\right) - \Lambda\beta^3\right] = 0, \tag{45}$$

if the spacetime has an outer or cosmological event horizon. For $T_{ab} = 0$ in Eq. (45), we get Kerr-de Sitter solution [2]. We note that the assumption of integral 2-manifolds orthogonal to both the Killing fields ξ^a and ϕ^a was crucial to the proof. For a completely general stationary axisymmetric spacetime, the existence of such submanifolds is not guaranteed, and thus an outer horizon may not exist in such cases, even for $\Lambda > 0$.

To summarize, we have found that in both the static and the stationary axisymmetric cases, existence of an outer Killing horizon requires a violation of the strong energy condition. This can be through a positive cosmological constant, for which there is strong observational evidence, or through exotic matter.

Acknowledgement

The authors wish to thank the anonymous referees for insightful questions and comments. S. B. also wishes to thank A. Ghosh for useful discussions. S. B.'s work was supported by a fellowship from his institution SNBNCBS.

[1] D. Kastor and J. H. Traschen, Phys. Rev. D **47**, 5370 (1993).
[2] B. Carter, Commun. Math. Phys. **10**, 280 (1968).
[3] A. G. Riess *et al.* [Supernova Search Team Collaboration], Astron. J. **116**, 1009 (1998).
[4] S. Perlmutter *et al.* [Supernova Cosmology Project Collaboration], Astrophys. J. **517**, 565 (1999).
[5] G. W. Gibbons and S. W. Hawking, Phys. Rev. D **15**, 2738 (1977).
[6] A. Lahiri, Mod. Phys. Lett. A **8**, 1549 (1993).
[7] R. M. Wald, "General Relativity," *Chicago, Usa: Univ. Pr. (1984).*
[8] E. Gourgoulhon and J. L. Jaramillo, Phys. Rept. **423**, 159 (2006).
[9] S. W. Hawking and G. F. R. Ellis, "The Large scale structure of space-time," *Cambridge University Press, Cambridge, (1973).*
[10] B. Carter, J. Math. Phys. **10**, 70 (1969).